Chapter 5: Equations

Equation 5.1:

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

Equation 5.2:

$$\mathrm{E}(Z) = \sum_{j=1}^{m} z_j \mathrm{P}(z_j)$$

Equation 5.3:

$$\mathrm{V}(Z) = \sum_{j=1}^{m} \left(z_j - \mathrm{E}(Z)\right)^2 \mathrm{P}(z_j)$$

Equation 5.4:

$$\mathrm{V}(Z) \equiv \mathrm{E}\left(\left(Z - \mathrm{E}(Z)\right)^2\right)$$

Equation 5.5:

$$\mathrm{E}(\varepsilon_i) = 0$$

Equation 5.6:

$$\mathrm{V}(\varepsilon_i) = \mathrm{E}\left(\varepsilon_i - \mathrm{E}(\varepsilon_i)\right)^2 = \mathrm{E}(\varepsilon_i)^2 = \sigma^2$$

Equation 5.7:

$$\mathrm{COV}(W,Z)=\sum_{j=1}^{m}\sum_{k=1}^{n}\left(w_j-\mathrm{E}(W)\right)\left(z_k-\mathrm{E}(Z)\right)\mathrm{P}\left(w_j,z_k\right)$$

Equation 5.8:

$$\mathrm{COV}(W,Z)=\mathrm{E}(V)=\mathrm{E}\left[\left(W-\mathrm{E}(W)\right)\left(Z-\mathrm{E}(Z)\right)\right]$$

Equation 5.9:

$$\mathrm{COV}\left(\varepsilon_i,\varepsilon_j\right)=\mathrm{E}\left[\left(\varepsilon_i-\mathrm{E}\left(\varepsilon_i\right)\right)\left(\varepsilon_j-\mathrm{E}\left(\varepsilon_j\right)\right)\right]$$

Equation 5.10:

$$\mathrm{COV}(\varepsilon_i,\varepsilon_j)=\mathrm{E}\left[\left(\varepsilon_i-\mathrm{E}\left(\varepsilon_i\right)\right)\left(\varepsilon_j-\mathrm{E}\left(\varepsilon_j\right)\right)\right]=\mathrm{E}(\varepsilon_i\varepsilon_j)$$

Equation 5.11:

$$\mathrm{COV}\left(\varepsilon_i,\varepsilon_j\right)=0$$

Equation 5.12:

$$\mathrm{E}(y_i)=\mathrm{E}(\alpha+\beta x_i+\varepsilon_i)$$

Equation 5.13:

$$\mathrm{E}\left(\sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{n} \mathrm{E}\left(y_i\right)$$

Equation 5.14:

$$\mathrm{E}\left(y_i\right) = \mathrm{E}\left(\alpha\right) + \mathrm{E}\left(\beta x_i\right) + \mathrm{E}\left(\varepsilon_i\right)$$

Equation 5.15:

$$\mathrm{E}\left(\alpha\right) = \alpha$$

Equation 5.16:

$$\mathrm{E}\left(y_i\right) = \alpha + \beta x_i$$

Equation 5.17:

$$y_i = \mathrm{E}\left(y_i\right) + \varepsilon_i$$

Equation 5.18:

$$y_i - \mathrm{E}\left(y_i\right) = \left(\alpha + \beta x_i + \varepsilon_i\right) - \left(\alpha + \beta x_i\right) = \varepsilon_i$$

Equation 5.19:

$$\mathrm{V}\left(y_i\right) = \mathrm{E}\left(y_i - \mathrm{E}\left(y_i\right)\right)^2 = \mathrm{E}\left(\varepsilon_i^2\right)$$

Equation 5.20:

$$\mathrm{V}\left(y_i\right)=\mathrm{E}\left(\varepsilon_i^2\right)=\sigma^2$$

Equation 5.21:

$$\mathrm{COV}\left(y_i,y_j\right)=\mathrm{E}\left(\varepsilon_i\varepsilon_j\right)$$

Equation 5.22:

$$\mathrm{COV}\left(y_i,y_j\right)=\mathrm{COV}\left(\varepsilon_i,\varepsilon_j\right)=0$$

Equation 5.23:

$$b=\frac{\sum_{i=1}^{n}(x_i-\overline{x})y_i}{\sum_{i=1}^{n}(x_i-\overline{x})x_i}$$

Equation 5.24:

$$b=\frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)\left(\alpha+\beta x_i+\varepsilon_i\right)}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}$$

Equation 5.25:

$$b=\alpha\frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}+\beta\frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}+\frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)\varepsilon_i}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}$$

Equation 5.26:

$$\alpha \frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}=\alpha\frac{0}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}=0$$

Equation 5.27:

$$\beta \frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}=\beta$$

Equation 5.28:

$$b=\beta+\frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)\varepsilon_i}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}$$

Equation 5.29:

$$\frac{\sum_{i=1}^{n}(x_i-\overline{x})\varepsilon_i}{\sum_{i=1}^{n}(x_i-\overline{x})x_i}=\frac{\sum_{i=1}^{n}(x_i-\overline{x})(\varepsilon_i-\overline{\varepsilon})}{\sum_{i=1}^{n}(x_i-\overline{x})^2}\frac{1/(n-1)}{1/(n-1)}=\frac{\mathrm{COV}(x_i,\varepsilon_i)}{\mathrm{V}(x_i)}$$

Equation 5.30:

$$b=\beta+\frac{\mathrm{COV}(x_i,\varepsilon_i)}{\mathrm{V}(x_i)}$$

Equation 5.31:

$$\text{bias} = \mathrm{E}(d) - \delta$$

Equation 5.32:

$$\mathrm{E}\left(b\right) = \mathrm{E}\left(\beta + \frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)\varepsilon_i}{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)x_i} \right)$$

Equation 5.33:

$$\mathrm{E}\left(b\right) = \mathrm{E}\left(\beta\right) + \mathrm{E}\left(\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)\varepsilon_i}{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)x_i} \right)$$

Equation 5.34:

$$\mathrm{E}\left(ky_i\right) = k\ \mathrm{E}\left(y_i\right)$$

Equation 5.35:

$$\mathrm{E}\left(\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)\varepsilon_i}{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)x_i} \right) = \frac{1}{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)x_i} \mathrm{E}\left(\sum_{i=1}^{n}\left(x_i - \overline{x}\right)\varepsilon_i \right)$$

Equation 5.36:

$$\frac{1}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}\sum_{i=1}^{n}\left(\mathrm{E}\left(x_i-\overline{x}\right)\varepsilon_i\right)=\frac{1}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)\mathrm{E}\left(\varepsilon_i\right)$$

Equation 5.37:

$$\mathrm{E}\left(b\right)=\mathrm{E}\left(\beta\right)+\mathrm{E}\left(\frac{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)\varepsilon_i}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)x_i}\right)=\beta+0=\beta$$

Equation 5.38:

$$b=\frac{\sum_{i=1}^{n}(x_1-\overline{x})y_i}{\sum_{i=1}^{n}(x_i-\overline{x})x_i}$$
$$=\frac{(x_1-\overline{x})}{\sum_{i=1}^{n}(x_i-\overline{x})x_i}y_1+\frac{(x_2-\overline{x})}{\sum_{i=1}^{n}(x_i-\overline{x})x_i}y_2+\cdots+\frac{(x_n-\overline{x})}{\sum_{i=1}^{n}(x_i-\overline{x})x_i}y_n$$

Equation 5.39:

$$\mathrm{E}\left(\overline{y}\right)=\mathrm{E}\left(\frac{\sum_{i=1}^{n}y_i}{n}\right)=\frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n}y_i\right)=\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(y_i\right)$$

Equation 5.40:

$$\mathrm{E}\left(\overline{y}\right)=\frac{1}{n}\left(n\alpha+\beta\sum_{i=1}^{n}x_i\right)=\alpha+\beta\overline{x}$$

Equation 5.41:

$$\mathrm{E}(a) = \mathrm{E}(\overline{y}) - \mathrm{E}(b\overline{x}) = (\alpha + \beta\overline{x}) - \overline{x}\ \mathrm{E}(b)$$

Equation 5.42:

$$\mathrm{E}(a) = (\alpha + \beta\overline{x}) - \overline{x}\beta = \alpha$$

Equation 5.43:

$$\mathrm{V}(ky_i) = k^2\mathrm{V}(y_i)$$

Equation 5.44:

$$\mathrm{V}\left(\sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{n} \mathrm{V}(y_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{i} \mathrm{COV}(y_i, y_j)$$

Equation 5.45:

$$\mathrm{V}\left(\sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{n} \mathrm{V}(y_i)$$

Equation 5.46:

$$\mathrm{V}\left(\sum_{i=1}^{n} k_i y_i\right) = \sum_{i=1}^{n} k_i^2 \mathrm{V}(y_i)$$

Equation 5.47:

$$b = \frac{(x_1 - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} y_1 + \frac{(x_2 - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} y_2 + \cdots + \frac{(x_n - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} y_n$$

Equation 5.48:

$$\mathrm{V}(b) = \left(\frac{(x_1 - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} \right)^2 \mathrm{V}(y_1) + \left(\frac{(x_2 - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} \right)^2 \mathrm{V}(y_2) + \cdots + \left(\frac{(x_n - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} \right)^2 \mathrm{V}(y_n)$$

Equation 5.49:

$$\begin{aligned} \mathrm{V}(b) &= \frac{\sigma^2}{\left(\sum_{i=1}^{n}(x_i - \overline{x})^2 \right)^2} \left((x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2 \right) \\ &= \frac{\sigma^2}{\left(\sum_{i=1}^{n}(x_i - \overline{x})^2 \right)^2} \left(\sum_{i=1}^{n}(x_i - \overline{x})^2 \right) \end{aligned}$$

Equation 5.50:

$$\mathrm{V}(b) = \frac{\sigma^2}{\left(\sum_{i=1}^{n}(x_i - \overline{x})^2 \right)^2} \left(\sum_{i=1}^{n}(x_i - \overline{x})^2 \right) = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

Equation 5.51:

$$\mathrm{V}(a) = \sigma^2 \left(\frac{1}{n} + \frac{\overline{x}^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2} \right)$$

Equation 5.52:

$$d = \left(\frac{(x_1 - \overline{x})}{\sum_{i=1}^{n} (x_i - \overline{x})^2} + z_1 \right) y_1 + \left(\frac{(x_2 - \overline{x})}{\sum_{i=1}^{n} (x_i - \overline{x})^2} + z_2 \right) y_2 + \cdots + \left(\frac{(x_n - \overline{x})}{\sum_{i=1}^{n} (x_i - \overline{x})^2} + z_n \right) y_n$$

Equation 5.53:

$$\beta = \mathrm{E}(d) = \mathrm{E}(b + c) = \mathrm{E}(b) + \mathrm{E}(c) = \beta + \mathrm{E}(c)$$

Equation 5.54:

$$0 = \mathrm{E}\left(\alpha \sum_{i=1}^{n} z_i + \beta \sum_{i=1}^{n} z_i x_i + \sum_{i=1}^{n} z_i \varepsilon_i \right) = \alpha \sum_{i=1}^{n} z_i + \beta \sum_{i=1}^{n} z_i x_i + \sum_{i=1}^{n} z_i \mathrm{E}(\varepsilon_i)$$

Equation 5.55:

$$\sum_{i=1}^{n} z_i = 0 \text{ and } \sum_{i=1}^{n} z_i x_i = 0$$

Equation 5.56:

$$\mathrm{V}(c) = \mathrm{V}\left(\sum_{ii=1}^{n} z_i y_i \right) = \sum_{i=1}^{n} z_i^2 \mathrm{V}(y_i) = \sigma^2 \sum_{i=1}^{n} z_i^2$$

Equation 5.57:

$$\mathrm{V}\left(d\right)=\sigma^{2}\sum_{i=1}^{n}\left(\frac{(x_i-\overline{x})}{\sum_{i=1}^{n}(x_i-\overline{x})^{2}}+z_i\right)^{2}$$

Equation 5.58:

$$\mathrm{V}(d)=\sigma^{2}\frac{\sum_{i=1}^{n}(x_i-\overline{x})^{2}}{\left(\sum_{i=1}^{n}(x_i-\overline{x})^{2}\right)^{2}}+\sigma^{2}\sum_{i=1}^{n}z_i^{2}+2\sigma^{2}\frac{\sum_{i=1}^{n}(x_i-\overline{x})z_i}{\sum_{i=1}^{n}(x_i-\overline{x})^{2}}$$

Equation 5.59:

$$\mathrm{V}\left(d\right)=\frac{\sigma^{2}}{\sum_{i=1}^{n}(x_i-\overline{x})^{2}}+\sigma^{2}\sum_{i-1}^{n}z_i^{2}+2\sigma^{2}\frac{\sum_{i=1}^{n}x_iz_i-\overline{x}\sum_{i=1}^{n}z_i}{\sum_{i=1}^{n}(x_i-\overline{x})^{2}}$$

Equation 5.60:

$$\mathrm{V}(d)=\mathrm{V}(b)+\mathrm{V}(c)\geq\mathrm{V}(b)$$